A Generalization of the Direct-Stiffness Method of Structural Analysis

Robert E Jones*
The Boeing Company Seattle, Wash

This paper presents a generalization of the direct-stiffness method of structural analysis which is expected to be of particular value in the analysis of plate and shell bending problems The methods given eliminate the difficulties encountered in obtaining continuity of deflection and slope between neighboring elements in the structure idealized by a finite element representation This is accomplished by formulating the method as a variational problem, thereby gaining for it some of the advantages of a stationary potential energy formulation, and by introducing Lagrange multiplier functions after the manner of E Reissner and K Washizu The plane stress and Kirchhoff plate bending problems are considered

Nomenclature

$\epsilon_{xx},\epsilon_{yy},\epsilon_{xy}$	= strains referred to Cartesian coordinates x, y
$\sigma_{xx},\sigma_{yy},\sigma_{xy}$	= stresses referred to Cartesian coordinates x, y
α,β	= Lagrange multiplier functions
u,v	= displacements referred to Cartesian coordinates x, y
t	= plate thickness
ν	= Poisson's ratio
A	= area of plate
S_I	= internal boundary lines of elements in idealized structures
S_E	= external boundary lines of elements in idealized structures
E	= Young's modulus
$\bar{u},\bar{v},\bar{w}$	= prescribed boundary displacements
X_n,Y_n	= resultant x, y direction stresses on an element boundary
$\bar{X}_n,\bar{Y}_n$	= prescribed boundary values of X, Y
n	= outward normal of a side of an element
X,Y	= body forces
X_n,Y_n	= line loads on internal element boundaries
θ_n	= bending slope of outward normal of element side having normal n, positive upward
w	= upward plate deflection normal to its surface
$M_x,M_y,M_{xy},$ Q_x,Q_y	= plate moments and shears
$\bar{Q}_n$	= applied upward line load on element boundary having normal n
$\bar{M}$	= applied line moment on element boundary line having normal n, positive when compressing upper surface
p	= upward surface load
P_m	= upward concentrated loads at nodes
$w_{,xx},w_{,yy},w_{,xy}$	= $\partial^2w/\partial x^2$, $\partial^2w/\partial y^2$, $\partial^2w/\partial x\partial y$

Introduction

In the analysis of complex structures, the use of finite element representations has proved very useful in recent years Such procedures involve the breakdown of the structure into large numbers of small, interconnected elements and the representation of the state of deformation and stress within the elements by forms that are simple compared to the state of the structure as a whole The analysis problem is transformed from one of determining the complicated mathematical functions pertinent for the continuum problem to one of determining the amplitudes of the various simple stress or deformation forms assumed for the elements of the idealized structure Although this latter problem involves much numerical calculation, the work consists mainly of matrix manipulation, so that it is conveniently carried out by digital computer

The two requirements on which are based the finite element developments are that 1) the elements are connected together in such a way that no discontinuities of deformation occur, and 2) the elements are in equilibrium subject to the external loads and the forces they exert on each other Satisfaction of these conditions is generally approximate Customarily, one of them is required explicitly (though not necessarily exactly), whereas the other follows implicitly according to the mechanics of the particular method being used In the "direct-stiffness method,"[1, 2] continuity of deformation is an explicit requirement, whereas the equilibrium requirement is implicit in the procedures of the method Attempts to extend the direct-stiffness method have occasionally met with perplexing difficulties A basic investigation aimed at explaining these difficulties has yielded two principal conclusions First, the method is actually based on the stationary total potential energy principle and is an application of the Ritz procedure for solving such minimal problems Second, a generalization of the method can be derived through the use of a broader variational principle, along the lines of the work of Reissner[3] and Washizu[4] The first item has since been studied by several researchers; its implications concerning the theory and practice of the direct-stiffness method are very important The second item provides a means of extending the direct-stiffness method to problems in which the deflection continuity requirements of the Ritz procedure appear difficult or impossible to satisfy with the finite element representation It forms the main subject of this paper

The variational formulations offer nearly foolproof procedures for deriving finite element analysis techniques They avoid a number of pitfalls inherent in the less rigorous formulations and have the added advantages of permitting statements to be made regarding bounds on solution quantities and convergence to exact solutions with vanishing element size [5–8]

General Discussion

The contrast between the usual Ritz procedure and the direct-stiffness method lies in the choice of displacement shapes of the latter Rather than the smooth deflection forms of the Ritz method, each extending over the entire structure, the direct stiffness method uses many localized displacement states, each restricted to a small part of the structure Overlapping of adjacent localized shapes is provided to insure

Received August 26, 1963; revision received February 6, 1964 The assistance of a number of associates is acknowledged The author is particularly grateful to M J Turner of The Boeing Company for encouragement and advice

* Research Specialist

sufficient generality of their total linear combination The convenient handling of such deflection shapes is accomplished through the small element structural representation The localized deflections are formed by the mutually compatible deformations of small groups of adjacent elements

Consider Fig 1, representing an array of plane elements Suppose the shaded region is to be deformed This can be accomplished by a movement of node 3, with nodes 1, 2, 4, 5, 6 held fixed Sides 1-4, 4-5, and 5-6 must also be held fixed, but the interior sides must deform to accommodate the movement of node 3 Further deformations can be accomplished by deflections of the interior sides For the sake of independence, these are accomplished one at a time, and node 3 and all nodes and sides exterior to the group are held fixed It is noted in passing that the overlap of localized shapes previously referred to is accomplished by subjecting the element groups surrounding nodes 1, 4, 5, 6, and 2 to similar deformations, and so on for every node throughout the structure

For convenience in digital computation, the deformations of single elements rather than those of element groups are prescribed The digital machine is instructed to combine properly the element deformations in order to form those of the groups This procedure implies certain requirements on the deflection shapes attributed to each element Consider, for example, element 1-2-3 in the figure Its available deflection patterns must permit the movement of one node at a time and one side at a time Moreover, the precise shapes of its side motions taken individually, as well as those due to node movement, must be identical to those for all other elements, regardless of their form or orientation Without such element deformation characteristics, deflection continuity could not be had This discussion is not limited to simple arrays of triangular elements For example, elements 3-4-5 and 3-5-6 could be replaced by a single quadrilateral element Much more complicated array and element forms could be considered also, and the same independence requirements on the individual element deformations would hold true

There are situations in which the element deformation shapes cannot be chosen so as to provide the independence requirements described This occurs in the Kirchhoff plate bending problem, where both deflection and bending slope continuity are required There are other situations in which it is very difficult, though not impossible, to provide the independencies To handle such cases while adhering to the localized deformation concept of the direct-stiffness method, a basic extension of the method is needed It is necessary to remove certain deflection continuity requirements from their explicit status in the digital machine instructions and

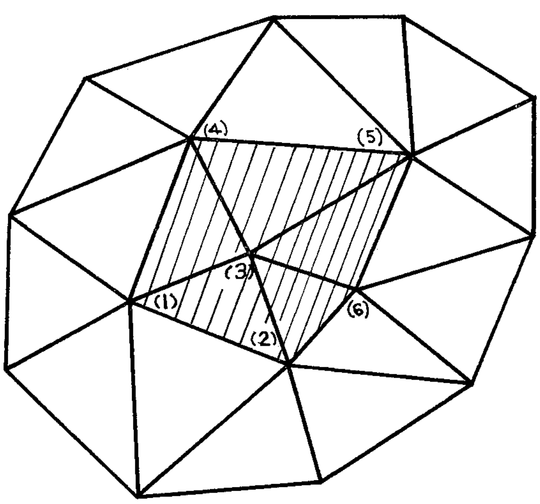

Fig 1 Array of triangular elements in plane deformation

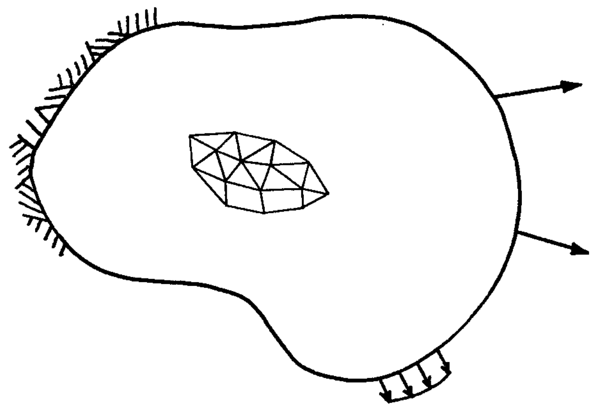

Fig 2 Plane region and partial elemental breakdown

to replace them with an implicit status similar to that of the equilibrium requirements themselves The solution of prob lems by such a method will in the end provide deflection continuity In some cases it will be exact, whereas in others, at the discretion of the analyst, the deflections will be continuous in a weighted average sense

Variational Formulation of Plane Stress Problem

The plane region shown in Fig 2 is cut by various lines into a large number of small elements The lines need not be straight for present purposes, and the elements may be of any shape The purpose of this section is to formulate a general variational procedure applicable to the structural analysis of this idealized structure Displacement discontinuities and nonequilibrated stress discontinuities will be permitted to occur across common element boundaries The variational principle of Reissner[3] is adopted as a starting point with various requirements added to it in order to cope with the present problem It will be shown that the variation of the functional given in Eq (1) yields all the conditions needed to qualify this small element representation as an approximate solution method consistent with the laws of the plane stress theory of elasticity The quantities subjected to independent variation are u, v, σ_{xx}, σ_{yy}, σ_{xy}, α, β α and β are the Lagrange multiplier functions which serve to constrain the deformations such that displacement discontinuities across element boundaries do not occur The quantities X_n and Y_n are constrained to be consistent on element boundaries with the stresses It is noted that the stresses need not satisfy the conditions of equilibrium, the statical boundary conditions, nor the conditions of compatibility The displacements need not be continuous across element boundaries, nor satisfy the geometrical boundary conditions or the conditions of equilibrium The stress-displacement elastic law need not be satisfied Subscripts 1 and 2 refer to elements on either side of a common element boundary The functional is

$$\begin{aligned}\frac{\Pi}{t} = {} & \int_A\int \left[\sigma_{xx}\frac{\partial u}{\partial x} + \sigma_{yy}\frac{\partial v}{\partial y} + \sigma_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] dA - \\ & \frac{1}{2E}\int_A\int [(\sigma_{xx} + \sigma_{yy})^2 + 2(1+\nu)(\sigma_{xy}{}^2 - \sigma_{xx}\sigma_{yy})]dA - \\ & \int_{S_I}[\alpha(u_1 - u_2) + \beta(v_1 - v_2)]dS_I - \\ & \int_{S_{E\bar{u}}}\alpha(u - \bar{u})dS_E - \int_{S_{E\bar{v}}}\beta(v - \bar{v})dS_E - \\ & \int_A (Xu + Yv)dA - \int_{S_{E\bar{X}}}\bar{X}_n u\, dS_E - \int_{S_{E\bar{Y}}}\bar{Y}_n v\, dS_E - \\ & \int_{S_I}\left(X_{ns}\frac{u_1 + u_2}{2} + Y_n\frac{v_1 + v_2}{2}\right)dS_I \qquad (1)\end{aligned}$$

The external boundary S_E has been divided into the portions over which specific boundary conditions have been required; e g , $S_{E\bar{u}}$ is that portion of the external boundary over which the displacement $\bar{u}$ is prescribed The integrals over S_E extend over the entire exterior boundary of the region, while those over S_I extend over all internal element boundaries The integrals over A extend over the entire surface area of the region including the element boundaries Performing the variations, integrating terms like $\sigma_{xx}(\partial\delta u/\partial x)$ by parts, and taking the resulting line integrals to the element boundaries, it is found that

$$\frac{\delta\Pi}{t} = -\int_A\int\left[\left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + X\right)\delta u + \left(\frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{xy}}{\partial x} + Y\right)\delta v\right]dA + \int_A\int\left[\left\{\frac{\partial u}{\partial x} - \frac{\sigma_{xx} - \nu\sigma_{yy}}{E}\right\}\delta\sigma_{xx} + \left\{\frac{\partial v}{\partial y} - \frac{\sigma_{yy} - \nu\sigma_{xx}}{E}\right\}\delta\sigma_{yy} + \left\{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \frac{2(1+\nu)}{E}\sigma_{xy}\right\}\delta\sigma_{xy}\right]dA + \int_{S_{E\bar{X}}}(X_n - \bar{X}_n)\delta u\, dS_E + \int_{S_{E\bar{Y}}}(Y_n - \bar{Y}_n)\delta v\, dS_E - \int_{S_{E\bar{u}}}(u - \bar{u})\delta\alpha\, dS_E - \int_{S_{E\bar{v}}}(v - \bar{v})\delta\beta\, dS_E + \int_{S_I}\frac{X_{n1} + X_{n2} - X_{ns}}{2}\times\delta(u_1 + u_2)dS_I + \int_{S_I}\frac{Y_{n1} + Y_{n2} - Y_{ns}}{2}\delta(v_1 + v_2)dS_I - \int_{S_I}(u_1 - u_2)\delta\alpha\, dS_I - \int_{S_I}(v_1 - v_2)\delta\beta\, dS_I - \int_{S_I}\left[\left(\alpha - \frac{X_{n1} - X_{n2}}{2}\right)\delta(u_1 - u_2) + \left(\beta - \frac{Y_{n1} - Y_{n2}}{2}\right)\delta(v_1 - v_2)\right]dS_I - \int_{S_{E\bar{u}}}(\alpha - X_n)\delta u\, dS_E - \int_{S_{E\bar{v}}}(\beta - Y_n)\delta v\, dS_E \quad (2)$$

The completeness of the functional can be demonstrated The first integral requires that the stresses satisfy the differential equations of equilibrium The second requires that the elastic relationship between stress and displacement be satisfied The third through sixth integrals require satisfaction of boundary conditions on stress and displacement The seventh through tenth integrals require satisfaction of stress jump vs line-load equilibrium and deflection continuity, respectively, across common element boundaries The last three integrals serve to define α and β The functional is seen to provide all the requirements of the elasticity problem and, therefore, will serve as an acceptable formulation of the finite element analysis method

The retention in the functional of stresses which are independent of the displacements is a degree of generality which might have been omitted for present purposes It has been done in order to suggest a means of stress calculation (retention of independent stresses) which has not received consideration in application of finite element methods, and in order to permit identification of the present approach as one in between the potential and complementary energy formulations It is of some interest to demonstrate the latter fact In the first case, if the stresses are defined by the stress-displacement equations in terms of the displacements (the second integral becomes identically satisfied) and if the displacements are made to satisfy all continuity and geometrical boundary conditions identically, then there is obtained the stationary potential energy formulation, which consists of only the first, third, fourth, seventh, and eighth integrals Recognition of this fact is aided by reversing the integration by parts process and introducing the strain energy function expressed in terms of the strains It is worth noting that, by formulating the direct-stiffness method by the variational principle, it is found that it is unnecessary to require that the assumed element deformation states satisfy any equilibrium conditions at all, including, in particular, the differential equations of equilibrium inside an element

To reduce the general variational equation to the complementary energy principle, let the stresses satisfy the differential equations of equilibrium in each element, the stress-jump condition across element boundaries and the statical boundary conditions Further, integrate terms like $(\partial u/\partial x)\delta\sigma_{xx}$ by parts, and define the complementary potential

$$B = \frac{1}{2E}\left[(\sigma_{xx} + \sigma_{yy})^2 + 2(1+\nu)(\sigma_{xy}^2 - \sigma_{xx}\sigma_{yy})\right]$$

The result will be the complementary energy principle, which here contains implicitly, in addition to the usual compatibility and geometrical boundary condition requirements, the presently pertinent requirement of deflection continuity across common element boundaries

Functional for Direct-Stiffness Method

The present work is aimed primarily at a needed generalization of the direct-stiffness method of finite element analysis The formulation given here is a specialization of the equations of the previous section and satisfies the need for generalization

Suppose that the stresses are defined in terms of the displacements by the elastic laws Then the functional

$$\frac{\Pi}{t} = \int_A\int\Phi\, dA - \int_{S_I}[\alpha(u_1 - u_2) + \beta(v_1 - v_2)]dS_I - \int_{S_{E\bar{u}}}\alpha(u - \bar{u})dS_E - \int_{S_{E\bar{v}}}\beta(v - \bar{v})dS_E - \int_A\int(Xu + Yv)dA - \int_{S_{E\bar{X}}}\bar{X}_n u\, dS_E - \int_{S_{E\bar{Y}}}\bar{Y}_n v\, dS_E - \int_{S_I}\left[X_{ns}\frac{u_1 + u_2}{2} + Y_{ns}\frac{v_1 + v_2}{2}\right]dS_I \quad (3)$$

where

$$\Phi = \frac{E}{2(1-\nu^2)}\left[\epsilon_{xx}^2 + 2\nu\epsilon_{xx}\epsilon_{yy} + \epsilon_{yy}^2 + \frac{1-\nu}{2}\epsilon_{xy}^2\right]$$

yields the variational equation

$$\frac{\delta\Pi}{t} = \int_A\int\left[\frac{\partial\Phi}{\partial\epsilon_{xx}}\delta\epsilon_{xx} + \frac{\partial\Phi}{\partial\epsilon_{yy}}\delta\epsilon_{yy} + \frac{\partial\Phi}{\partial\epsilon_{xy}}\delta\epsilon_{xy}\right]dA - \int_{S_I}[(u_1 - u_2)\delta\alpha + (v_1 - v_2)\delta\beta]dS_I - \int_{S_I}[\alpha\delta(u_1 - u_2) + \beta\delta(v_1 - v_2)]dS_I - \int_{S_{E\bar{u}}}(u - \bar{u})\delta\alpha\, dS_E - \int_{S_{E\bar{v}}}(v - \bar{v})\delta\beta\, dS_E - \int_{S_{E\bar{u}}}\alpha\delta u\, dS_E - \int_{S_{E\bar{v}}}\beta\delta v\, dS_E - \int_A\int(X\delta u + Y\delta v)dA - \int_{S_{E\bar{X}}}\bar{X}_n\delta u\, dS_E - \int_{S_{E\bar{Y}}}\bar{Y}_n\delta v\, dS_E - \int_{S_I}\left(X_n\,\delta\frac{u_1 + u_2}{2} + Y_n\,\delta\frac{v_1 + v_2}{2}\right)dS_I \quad (4)$$

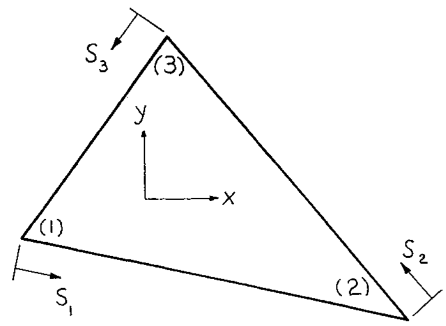

Fig 3 Triangular element and coordinate systems

which may be found to supply all the conditions needed to qualify it as a formulation consistent with the plane stress theory of elasticity The first integral yields the internal work done in the elements due to variations of their deformation parameters This is equivalent to the work done on the elements by their edge stresses, provided X and Y are zero; the latter viewpoint has often been used in formulations of the direct-stiffness method The second, fourth, and fifth integrals enforce continuity and satisfaction of geometrical boundary conditions The last four integrals account for the work done by the external loads The third, sixth, and seventh integrals supply the extra equations needed for the determination of α, β, which can be found to be the element edge stresses on S_E and the average of the element edge stresses at interior element boundaries In its implicit compatibility requirements and its retention of the stress unknowns α and β, the formulation is seen to have the character of both the potential and complementary energy methods The preceding equation is the one needed for the formulation of the algebraic equations of the generalized direct-stiffness method, which is the subject of the next section

Formulation of Algebraic Equations

Equation (4) yields, as a consequence of setting each of its independent variations to zero, the set of algebraic equations governing the structural analysis problem The choice of the freedoms of a single element provides the substance of these equations Suppose the deformation of a single representative element is given by

$$u = \sum_i a_i f_i(x,y)$$
$$v = \sum_i b_i g_i(x,y) \tag{5}$$

where f_i, g_i are assumed functions, and the a_i and b_i are unknown constants which are subjected to independent variation

Consider the first integral of Eq (4) for one element only On dropping the δa_i and δb_i, the following form results

$$\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ & & & \\ k_{n1} & \cdots & \cdots & k_{nn} \end{bmatrix} \begin{Bmatrix} a_i \\ \\ b_i \end{Bmatrix} \tag{6}$$

k_{ij} is a symmetric matrix which is called the stiffness matrix of the element; it is the central quantity of the direct-stiffness method Its derivation here is based on the strain energy function Φ The extension of the first integral to the entire area involves the constraints between the a_i and b_i of neighboring elements due to explicit continuity require ments If such explicit requirements are made, then the a_i and b_i of an element must be grouped in the required way with those of adjacent elements in order to obtain truly independent variations Each of the forms resulting from setting the resulting independent variations to zero will therefore contain terms from several elements The process of gathering together these contributions from several elements amounts to adding together the stiffnesses of adjacent elements against the deformations in which they participate mutually This is termed the "merge," and is difficult in proportion to the number of elements involved In the present case, where the explicit continuity requirements are largely relaxed, there is little dependence between the a_i and b_i of different elements and the mathematical forms arising from the independent variations in the first integral of Eq (4), each contain, largely, quantities associated with a single element The merge is therefore quite simple, much more so in fact than that of the ungeneralized direct-stiffness method

The functions α and β are also represented by assumed shapes, for example

$$\alpha = \sum_i c_i h_i(s)$$
$$\beta = \sum_i d_i k_i(s) \tag{7}$$

s denotes distance along an element boundary, measuring from one end Figure 3 shows a triangular element in which such quantities are designated for all sides h_i and k_i are assumed functions; c_i and d_i are unknown constants to be subjected to independent variation The choice of the functions h_i and k_i is based on the shapes of the deflection discontinuity between elements as functions of the side-length variables s This is predictable from the forms of the functions f_i and g_i in Eq (5)

The functions h_i and k_i may be chosen such that the variational formulation yields complete continuity in the final solution, or a less rigid requirement of weighted average continuity may be had by reducing the number of functions kept in Eq (7) The complete matrix statement of the algebraic equations arising from Eq (4) is given in Eq (8):

equations from $\delta a_i = 0$ $\delta b_i = 0$	Stiffness matrix k_{ij}		a_i b_i	=	F_i	
equations from $\delta c_i = 0$ $\delta d_i = 0$	Constraint matrix	All zero in this submatrix	c_i d_i		0 0	(8)

Integrals two, four, and five provide the lower left submatrix The calculation of the coefficients in this submatrix requires the determination of the side motions of an element associated with its a_i and b_i This may be done, and the integrations performed, for a single element at a time However, since the c_i and d_i belong to particular network lines (between nodes) rather than to the element themselves, a merging process is needed The fourth and fifth integrals, which contain $u - \bar{u}$ and $v - \bar{v}$, involve only the a_i and b_i of a single element The second integral contains $u_1 - u_2$ and $v_1 - v_2$, so that each of the rows of the submatrix pertaining to c_i, d_i on an interior network line contain the a_i and b_i from two elements—those common to the line The merge of the lower left submatrix, then, is very simple, usually considerably more so than that of the upper left submatrix The determination of the upper right submatrix follows the same lines discussed previously However, the entire matrix is symmetrical, so that this submatrix need not be calculated

The remaining integrals provide the external work terms, which furnish the right-hand column of the matrix equation Determination of these terms by the work integrals, as is done here, is an improvement over the method of taking all loads to the nodes as point loads The generality of the direct-stiffness method is thus seen to include point, line, and distributed loads

The equations arising from equating to zero the coefficients of δc_i and δd_i on all network lines are the ones which provide continuity of deflection between elements It is their presence that permits the independence of the a_i and b_i which makes the merge of the present method so simple The equations arising from equating to zero the coefficients of δa_i and δb_i contain as unknowns the c_i and d_i as well as a_i and b_i The total of the unknowns in the present method is larger than that of the direct-stiffness method, in some cases much larger On the other hand, the merge procedure, which is the core of the direct-stiffness method, is made much simpler by the generalizations discussed here In addition, the present formulation is particularly susceptible to solution by matrix partitioning, which simplifies the calculations of the solution

Variational Formulation of Plate Bending Problem

The plane region and elemental breakdown shown in Fig 2 is representative also of the plate bending problem; the loads are now considered to be applied out of plane Again, the lines need not be straight, and the elements may be of any shape Each element is presumed to have a simple state of curvature and twist The deformations are taken to be consistent with the Kirchhoff hypothesis In-plane deformations may be considered separately and are ignored here The concepts explained in previous sections concerning the assumed shapes and the variational formulation carry over to the present case

A variational formulation will be given for the case in which there may be displacement and slope discontinuities across common element boundaries Rather than start from a variational principle of the type of Reissner, it will be presumed to start that the bending and twisting moments are defined in terms of the curvatures and the twist by the usual equations of plate theory The Lagrange multipliers α and β are introduced in order to require, respectively, displacement and normal slope continuity across common element boundaries By normal slope θ_n, will be meant the derivative $\partial w/\partial n$, where n is the outward directed normal from an element boundary Similarly, M_n and Q_n are the plate bending moment and transverse shear on an element boundary with normal n The needed functional is

$$\pi = \int_A\int\left[M_x\left(\frac{\partial^2 w}{\partial x^2}\right) + M_y\left(\frac{\partial^2 w}{\partial y^2}\right) + 2M_{xy}\left(\frac{\partial^2 w}{\partial x\partial y}\right)\right]dA - \int_{S_I}\alpha(w_1 - w_2)dS_I - \int_{S_I}\beta(\theta_{n1} + \theta_{n2})dS_I - \int_{S_{E\bar{w}}}\alpha(w - \bar{w})dS_E - \int_{S_{E\bar{\theta}n}}\beta(\theta_n - \bar{\theta}_n)dS_E - \int_{S_I}\bar{Q}_n\frac{w_1 + w_2}{2}dS_I - \int_{S_I}\bar{M}_n\frac{\theta_{n1} - \theta_{n2}}{2}dS_I - \int_A\int pw\,dA - \int_{S_{E\bar{Q}n}}\bar{Q}_n w\,dS_E - \int_{S_{E\bar{M}n}}M_n\theta_n\,dS_E - \sum_m P_m w_m \quad (9)$$

where $\sum_m$ extends over all nodes at which concentrated loads act The quantities subjected to independent variation are α, β, and w Note that p, P_m, and $\bar{Q}_n$ are upward directed loads and that $\bar{M}_n$, the applied line moment, is plus when it produces compression in the upper surface of the no 1 element joining a cut (choice of no 1 and no 2 elements is arbitrary) On an external edge, $\bar{M}_n$ is plus when it produces compression in the upper surface of the element joining the edge On an internal edge, $\bar{Q}_n$ and $\bar{M}_n$ are divided equally between the elements common to the edge

Upon carrying out the variation, there is obtained

$$\delta\pi = \int_A\int\left[\frac{\partial\Phi}{\partial w_{xx}}\delta w_{xx} + \frac{\partial\Phi}{\partial w_{yy}}\delta w_{yy} + \frac{\partial\Phi}{\partial w_{xy}}\delta w_{xy}\right]dA - \int_{S_I}[(w_1 - w_2)\delta\alpha + (\theta_{n1} + \theta_{n2})\delta\beta]dS_I - \int_{S_I}[\alpha\delta(w_1 - w_2) + \beta\delta(\theta_{n1} + \theta_{n2})]dS_I - \int_{S_{E\bar{w}}}(w - \bar{w})\delta\alpha\,dS_E - \int_{S_{E\bar{\theta}n}}(\theta_n - \bar{\theta}_n)\delta\beta\,dS_E - \int_{S_{E\bar{w}}}\alpha\delta w_n\,dS_E - \int_{S_{E\bar{\theta}n}}\beta\delta\theta_n\,dS_E - \int_{S_I}\bar{Q}_n\delta\frac{w_1 - w_2}{2}dS_I - \int_{S_I}\bar{M}_n\delta\frac{\theta_{n1} - \theta_{n2}}{2}dS_I - \int_A\int p\delta w\,dA - \int_{S_{E\bar{Q}n}}\bar{Q}_n\delta w\,dS_E - \int_{S_{E\bar{M}n}}\bar{M}_n\delta\theta_n\,dS_E - \sum_m P_m\delta w_m \quad (10)$$

Φ is the strain energy function of plate bending expressed in terms of the curvatures w_{xx}, w_{yy}, and the twist w_{xy} The preceding equation is the one which is used in formulating the simultaneous equations of the generalized direct-stiffness method

After two integrations by parts of the first integral, collecting terms and defining M_n and Q (the plate moment and shear on an edge with normal n) in terms of M_x, M_y, M_{xy}, Q_x, Q_y, and performing various other manipulations, it is found that the preceding functional provides the following requirements:

1) Satisfaction of equilibrium conditions in the interior of each element

2) Balance of line load $\bar{Q}_n$ and line moment $\bar{M}_n$ by the work equivalents of jumps in transverse shear and moment across common element boundaries and by the work equivalents of the shears and moments themselves on external boundaries (The conditions on transverse shear include the twisting moment, as usual)

3) The finite transverse loads at element corners associated with the jump in twisting moment around the corners are balanced by the work equivalents of applied concentrated loads at the node points

4) Displacement and slope are continuous across common element boundaries and satisfy geometric boundary conditions, in each case, either exactly or approximately

5) The quantities α and β are found to be, respectively, the average of Q_n and the average of M_n at common element boundaries and the values themselves at external boundaries where geometric boundary conditions are specified

Numbers 1–4 are all the requirements of the plate bending problem Their representation, by the functional given, establishes its completeness and makes unnecessary any additional requirements In particular, the satisfaction of the differential equation of equilibrium in the interior of the element is unnecessary in the choice of element deformation shapes

Formulation of the Algebraic Equations

The use of Eq (10) to generate the algebraic equations of the plate bending problem follows along the lines discussed previously in the section on the plane stress problem The deflection shape of an element is assumed as is Eq (5):

$$w = \sum_i a_i f_i(x,y) \tag{11}$$

where the a_i are constants to be subjected to variation The first integral of Eq (10), taken for a single element, yields the symmetrical-stiffness matrix of the element in the form

$$[k_{ij}] \ \{a_i\} \tag{12}$$

As in the plane problem, extension of the integral to the entire area involves constraints between the a_i of neighboring elements due to continuities which may be explicitly required In the present case, the resulting merge turns out to be very simple due to the partial relaxation of the explicit continuity conditions

The functions α and β are again represented by Eq (7) The h_i and k_i are chosen with regard to the shapes of the deflection and normal slope discontinuities between elements, respectively, based on the functions f_i in Eq (11) As in the plane stress case, depending on the number of functions retained in (7), the final result may have complete or approximate continuity The merge process associated with the c_i and d_i is quite simple The complete matrix equation is in the form of Eq (8) with references to b_i and δb_i deleted The discussion of the various parts of the coefficient matrix given previously applies again to the present case

References

[1] Turner, M J , Clough, R W , Martin, H C , and Topp, L J , "Stiffness and deflection analysis of complex structures " J Aeronaut Sci **23,** 805–823 (1956)

[2] Turner, M J , "The direct stiffness method of structural analysis," AGARD Structures and Materials Panel Meeting, Aachen, Germany (September 17, 1959)

[3] Reissner, E , "On a variational theorem in elasticity, ' J Math Phys **29,** 90–95 (1950)

[4] Washizu, K , "On the variational principles of elasticity and plasticity," TR 25-18, Aeroelastic and Structures Research Lab , Massachusetts Institute of Technology (March 1955)

[5] Melosh, R , "Basis for derivation of matrices for the direct stiffness method," AIAA J **1,** 1631–1637 (1963)

[6] de Veubeke, B F , "Duality between displacement and equilibrium methods with a view to obtaining upper and lower bounds to static influence coefficients," *AGARD Structures and Materials Panel Meeting* (Pergamon Press, London), to be published

[7] Melosh, R , "Development of the stiffness method to define bounds on elastic behavior of structures," Ph D Thesis, Department of Civil Engineering, Univ of Washington, Seattle, Wash (June 1962)

[8] McLay, R W , "An investigation into the theory of the displacement method of analysis for linear elasticity," PhD Thesis, Dept of Engineering Mechanics, Univ of Wisconsin, Madison, Wis (August 1963)